

die FREDHOLMSche Determinante der Gl. (15) bedeutet und N die oben mit e^{-iF} bezeichnete Größe ist, woraus dann die FREDHOLMSche Lösung der Integralgleichung (15) folgt. Da bei unserem Verfahren derselbe Ausdruck für $S(1, 1', g)$ folgt wie bei MATTHEWS und SALAM, so sollte die entsprechende Reihenentwicklung nach Potenzen von g zu der FREDHOLMSchen Lösung führen. Dabei müßte eine Identität bestehen zwischen den entsprechenden Ko-

effizienten, die in beiden Fällen dieselben Kombinationen der nach den beiden Verfahren beziehungsweise berechneten Propagatorausdrücke enthalten. Man würde erwarten, daß dadurch die Identität der entsprechenden Propagatoren folgen würde. Jedoch dürfte eine eingehende Untersuchung der betreffenden Propagatoren auf Grund des hier vorgeschlagenen Verfahrens erforderlich sein, um diese Frage zu entscheiden.

Unimodular Matrices Homomorphic to Lorentz Transformations in $n \geq 2$ Spacelike Dimensions

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(Z. Naturforschg. 22 a, 1293—1299 [1967]; received 30 June 1967)

Dedicated to Professor PASCUAL JORDAN on the occasion of his 65th birthday

We generalize the usual homomorphism between 2×2 unimodular matrices $B(\Lambda)$ and restricted 4×4 LORENTZ matrices Λ to the case of one timelike and $n \geq 2$ spacelike dimensions. For every such n which is even (odd), this generalization associates homomorphically to each restricted (orthochronous) $(n+1)$ -dimensional LORENTZ matrix a set of $N \times N$ -dimensional unimodular matrices, where $N=2^{n/2}$ or $2^{(n-1)/2}$, depending on whether n is even or odd. In the case $n \geq 2$, we prove the theorem that, if $B(\Lambda)$ and $B(\Lambda^T)$ are any two such unimodular matrices associated with Λ and its adjoint Λ^T , respectively, then $B(\Lambda^T) = \omega B(\Lambda)^\dagger$, where ω is an N^{th} root of unity and \dagger means hermitean adjoint. We also prove that for $n > 3$ one can select these two unimodular matrices so that this equation holds with $\omega=1$, but that no such selection is possible for $n=2, 3$.

The purpose of the present paper is to establish a homomorphic correspondence between a subgroup of the unimodular matrices and the LORENTZ transformations in $(n+1)$ -dimensional space with one timelike and $n \geq 2$ spacelike dimensions. This correspondence, which generalizes the well-known correspondence for $n=3$ between 2×2 unimodular matrices and 4×4 restricted LORENTZ transformations, is established in Section 1. We have devoted Section 2 to two proofs, which we believe to be simple, of a theorem on the unimodular matrix associated with the transpose of an $(n+1)$ -dimensional LORENTZ matrix for $n \geq 2$. This theorem is essen-

tially known for the case of three spatial dimensions^{1, 2}, the only accessible proof being, however, that in reference².

1. Homomorphism Between Lorentz Transformations and Unimodular Matrices for $n \geq 2$

Consider a fixed irreducible set τ_k ($k=1, 2, \dots, n$) of $n \geq 2$ hermitean matrices which satisfy the equation³

$$\tau_k \tau_l + \tau_l \tau_k = 2 \delta_{kl} 1 \quad (k, l = 1, 2, \dots, n). \quad (1)$$

¹ See, for example, p. 41, Eq. (2.8) of the article of E. P. WIGNER on "Unitary Representations of the Inhomogeneous LORENTZ Group Including Reflections" in Group Theoretical Concepts and Methods in Elementary Particle Physics, Gordon and Breach, New York 1964.

² I. M. GEL'FAND, R. A. MINLOS, and Z. YA. SHAPIRO, Representations of the Rotation and LORENTZ Groups and Their Applications, Pergamon Press, New York 1963, p. 170—171.

³ Even if not stated explicitly, the inequality $n \geq 2$ is always supposed to hold in this paper. Lower case Greek and Latin indices run over $0, 1, \dots, n$ and $1, 2, \dots, n$, respectively, and all equations involving free indices of these two types should be understood to hold for all values in these respective ranges. Summations over Greek and Latin indices also run over the full ranges of these indices.



It is well known⁴ that the τ_k and their products with positive and negative signs, are homomorphic to an abstract group of order 2^{n+1} . Exploiting this fact, one infers, in the first place, that for even n there is exactly one irreducible set $\{\tau_k\}$ ($k=1, 2, \dots, n$) within equivalence, the dimension of these matrices being $2^{n/2}$. Secondly, one concludes that there are precisely two such inequivalent irreducible sets for the case of odd n , the corresponding τ_k having dimension $2^{(n-1)/2}$. In the latter case, $\tau_n = \lambda_n \tau_1 \tau_2 \dots \tau_{n-1}$ in one of these irreducible sets and $\tau_n = -\lambda_n \tau_1 \tau_2 \dots \tau_{n-1}$ in the other one, where $\lambda_n = 1(i)$ if n is of the form $4l+1$ ($4l+3$), l being zero or a positive integer. If we chose the τ_k to be unitary, which is always possible because they generate a representation of a finite group, then the τ_k are also hermitean, since their squares are unity. This hermiticity property will be supposed to hold in what follows.

For each $(n+1)$ -vector (x^0, x^1, \dots, x^n) we define the $N \times N$ matrix $X(x)$ as the linear combination

$$X(x) = \sum_{\mu} x^{\mu} \tau_{\mu} \quad (2)$$

of the matrices $\tau_0, \tau_1, \dots, \tau_n$; τ_0 being the $N \times N$ unit matrix⁴.

The symbol $A = \|A^{\mu}_{\nu}\|$ ($\mu, \nu=0, 1, \dots, n$) will denote the matrix of a transformation leaving the form $(x^0)^2 - (x^1)^2 - \dots - (x^n)^2$ invariant, so that

$$A G A^T = G, \quad (3)$$

where $G = \|g_{\mu\nu}\|$ is a diagonal matrix with $g_{00}=1$, $g_{11}=\dots=g_{nn}=-1$ and T means transpose. In particular, $(A^T)^{\mu}_{\nu} = A^{\nu}_{\mu}$. Moreover, A will always refer either to an orthochronous LORENTZ transformation ($A^0_0 > 0$) or to a restricted one ($A^0_0 > 0$ and $\det A = 1$).

The $N \times N$ matrix $B(A)$ is defined as a unimodular solution of the equation

$$X(x') = X(Ax) = B(A) X(x) B(A)^{\dagger}, \quad (4)$$

for every A belonging to the groups of LORENTZ matrices, to be specified below, for which such solutions exist. Here † denotes hermitean adjoint and

$$x'^{\mu} = (Ax)^{\mu} = \sum_{\nu} A^{\mu}_{\nu} x^{\nu}.$$

⁴ P. JORDAN and E. P. WIGNER, Z. Phys. **47**, 631 [1928], especially pages 650 and 651. Although only the case of even n is explicitly considered in this reference, the methods of the cited two pages are easily applied to the case of odd $n \geq 3$.

Eq. (4) is supposed to hold identically in the x^{ν} . Hence the familiar argument using the linear independence of the τ_k shows that (4) can be regarded as a shorthand for the $n+1$ equations

$$\sum_{\mu} A^{\mu}_{\nu} \tau_{\mu} = B(A) \tau_{\nu} B(A)^{\dagger}. \quad (5)$$

The matrices A for which (4) or (5) admits a unimodular solution are given by the two theorems, valid for $n \geq 2$, which we now state.

Theorem 1. If n is even (odd), then $B(A)$ exists for each real orthochronous (restricted) $(n+1)$ -dimensional LORENTZ matrix A and is determined by (5) up to a multiplicative constant equal to an arbitrary N^{th} root of unity.

That the matrices A in Theorem 1 are the only ones for which such unimodular solutions $B(A)$ exist is implied by

Theorem 2. If B is an arbitrary unimodular N -dimensional matrix having the property that all the $B \tau_{\nu} B^{\dagger}$ are linear combinations

$$B \tau_{\nu} B^{\dagger} = \sum_{\mu} L^{\mu}_{\nu} \tau_{\mu} \quad (5')$$

of the τ_{μ} , then the unique matrix $\|L^{\mu}_{\nu}\|$ corresponding to B in the sense of (5') is a real $(n+1)$ -dimensional orthochronous (restricted) LORENTZ matrix for even (odd) n .

The set B_n of all matrices associated in the sense of (5') [or (5)] with the group of orthochronous or restricted LORENTZ matrices in the respective cases of n even or n odd forms a group, as follows easily with the aid of (5'). Theorems 1 and 2 inform us that there is a 1-to- N onto mapping of the pertinent one of these two LORENTZ groups onto B_n . Just as in the well known case $n=3$, this onto mapping is homomorphic because, if the unimodular matrices B_1 and B_2 correspond to the LORENTZ matrices A_1 and A_2 , respectively, then it is clear, for example from (4), that $B_1 B_2$ corresponds to $A_1 A_2$.

The irreducibility of the set τ_1, \dots, τ_n entails that of B_n . In fact, $\det \tau_k = -1$ for $n=2, 3$, as is well known, and $\det \tau_k = 1$ for $n > 3$, as follows from Eq. (17) of the present paper and the evenness of $N/2$ for $n > 3$. Since the τ_k are two-dimensional if $n=2, 3$, we therefore see that the matrices $i \tau_k$ are unimodular for these two values of n . Moreover, for $n \geq 2$ these matrices satisfy the equations

$$(i \tau_k) \tau_{\nu} (i \tau_k)^{\dagger} = \tau_k \tau_{\nu} \tau_k = \mp \tau_{\nu}$$

because of the hermiticity of the τ_k and (1). Hence we conclude that the obviously irreducible set $\{i \tau_k\}$

($k=1, \dots, n$) is a subset of B_n if $n=2, 3$, and hence that B_2 and B_3 are irreducible. Similarly, if $n>3$, one can show that the irreducible set $\{\tau_k\}$ of n unimodular matrices τ_k is in B_n , so that B_n is irreducible for such n . Conversely, the irreducibility of B_n entails that of $\{\tau_k\}$ for $n \geq 2$.

Let us now prove the stated theorems.

Proof of Theorem 1.

We begin by showing that $B(\mathcal{A})$ exists for each restricted LORENTZ matrix \mathcal{A} for $n \geq 2$.

Every such \mathcal{A} of $n+1$ dimensions can be written as a product

$$\mathcal{A} = R_2 \mathcal{A}_k R_1, \quad (6)$$

where R_1 and R_2 are proper spatial rotations and \mathcal{A}_k is a pure LORENTZ transformation along one of the spatial axes x^k ($k=1, 2, \dots, n$). It suffices to show that (5) has a unimodular solution for each LORENTZ matrix which is of one of these two types to conclude that (5) also has such a solution for any given restricted \mathcal{A} , since because of (6) and a previous remark the unimodular matrix

$$B(R_2) B(\mathcal{A}_k) B(R_1)$$

corresponds to this \mathcal{A} in the sense of (5).

We first prove that to the pure LORENTZ transformation

$$\begin{aligned} x'^0 &= \cosh \chi x^0 + \sinh \chi x^k, \\ x'^k &= \sinh \chi x^0 + \cosh \chi x^k, \\ x'^l &= x^l \quad (l \neq 0, k), \end{aligned} \quad (7)$$

with an arbitrary χ , there corresponds the unimodular matrix

$$B_k = \cosh \frac{1}{2} \chi \tau_0 + \sinh \frac{1}{2} \chi \tau_k \quad (8)$$

in the sense of (5).

To establish the unimodularity of B_k , we observe that Eqs. (1) state that each τ_k has unit square and anticommutes with each τ_l with $l \neq k$. Therefore, if this inequality obtains, $\tau_l = -\tau_k \tau_l \tau_k = -\tau_k \tau_l \tau_k^{-1}$, whence

$$\text{Trace } \tau_l = -\text{Trace } \tau_k \tau_l \tau_k^{-1} = -\text{Trace } \tau_l = 0.$$

Since $\tau_k^2 = 1$, the eigenvalues of τ_k can only be ± 1 . In view of the tracelessness of τ_k , half of these eigenvalues, i. e., $N/2$ of them, are equal to $+1$ and the remaining half are -1 . Consequently, because of the hermitean nature of τ_k , there exists a unitary matrix U_k such that

$$\tau_k = U_k \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{vmatrix} U_k^\dagger, \quad (9)$$

where 0 and 1 denote the $\frac{1}{2}N$ -dimensional zero and unit matrices, respectively. From (8), (9), and $\tau_0 = 1$,

$$\begin{aligned} B_k &= U_k \left\{ \cosh \frac{1}{2} \chi \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} + \sinh \frac{1}{2} \chi \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{vmatrix} \right\} U_k^\dagger \\ &= U_k \begin{vmatrix} e^{\chi/2} \cdot 1 & & 0 \\ & \ddots & \\ 0 & & e^{-\chi/2} \cdot 1 \end{vmatrix} U_k^\dagger, \end{aligned} \quad (10)$$

so that B_k is unitarily equivalent to a unimodular matrix and is therefore unimodular.

We proceed to show that B_k corresponds to \mathcal{A}_k . From (9), the hermiticity of B_k , and the fact that $\tau_0 = \tau_k^2 = 1$, we can obtain by direct calculation:

$$\begin{aligned} B_k \tau_0 B_k^\dagger &= B_k^2 = (\cosh \frac{1}{2} \chi \tau_0 + \sinh \frac{1}{2} \chi \tau_k)^2 \\ &= \cosh \chi \tau_0 + \sinh \chi \tau_k, \end{aligned} \quad (11a)$$

and similarly

$$B_k \tau_k B_k^\dagger = \sinh \chi \tau_0 + \cosh \chi \tau_k, \quad (11b)$$

$$B_k \tau_l B_k^\dagger = \tau_l \quad (l \neq k). \quad (11c)$$

From (11a) to (11c), B_k is seen to obey the $n+1$ Eqs. (5) for the pure LORENTZ transformation (7).

Next we deal with a rotation by any angle φ , which takes place in the plane containing the spatial axes x^k and x^l ($k \neq l$):

$$\begin{aligned} x'^k &= x^k \cos \varphi - x^l \sin \varphi, \\ x'^l &= x^k \sin \varphi + x^l \cos \varphi, \\ x'^\mu &= x^\mu \quad (\mu \neq k, l), \end{aligned} \quad (12)$$

We assert that the matrix

$$B_{kl} = \cos \frac{1}{2} \varphi \tau_0 + i \sin \frac{1}{2} \varphi i^{-1} \tau_k \tau_l \quad (13)$$

is unimodular and corresponds to this rotation.

That B_{kl} is unimodular can be shown by combining the facts that the matrix $i^{-1} \tau_k \tau_l$ ($k \neq l$) in (13) is hermitean and that it is also, because of (1), of unit square and traceless, with procedures parallel to those just employed in proving the unimodularity of B_k . Computations analogous to those carried out in the case of B_k serve to verify that B_{kl} is indeed a solution of (5) pertaining to the spatial rotation (12).

We now show that, for even n , (5) has a unimodular solution $B(\mathcal{A})$ for every orthochronous \mathcal{A} , and not merely for each restricted \mathcal{A} . This is in sharp contrast with the situation obtaining for odd $n \geq 3$, in which case (5) has such a unimodular solution only if \mathcal{A} is a restricted LORENTZ matrix, as will be proved later on in this section.

It is well known that each improper orthochronous \mathcal{A} can be written, for example, as the product

$$\mathcal{A} = i_n \mathcal{A}_0 \quad (14)$$

of the spatial reflection i_n :

$$\begin{aligned} x'^n &= -x^n, \\ x'^\mu &= x^\mu \quad (\mu=0, 1, \dots, n-1), \end{aligned} \quad (15)$$

and a restricted LORENTZ matrix \mathcal{A}_0 . Since $B(\mathcal{A}_0)$ exists, as is shown by our previous discussions, we only need to prove the existence of a unimodular matrix $B^{(n)}$ which corresponds to i_n for each even n to deduce that for every n of this type (5) possesses the unimodular solution $B(\mathcal{A}) = B^{(n)} B(\mathcal{A}_0)$ for every orthochronous \mathcal{A} . Such a $B^{(n)}$ is given by

$$\begin{aligned} B^{(2)} &= i \tau_1, \\ B^{(n)} &= \tau_1 \tau_2 \dots \tau_{n-1} \quad , \quad n \geq 4 \end{aligned} \quad (16)$$

for every even n ⁵.

To prove that the matrices $B^{(n)}$ in (16) are unimodular, one notices that

$$\det \tau_k = (-1)^{N/2}, \quad (17)$$

an equation which holds because $N/2$ of the eigenvalues of τ_k are 1 and the remaining $N/2$ are -1 . Combining (16) and (17) with the fact that $N/2$ is odd for $n=2, 3$ and even for $n \geq 4$, the asserted unimodularity property follows immediately.

For even n , one easily sees from (1) that the $B^{(n)}$ in (16) satisfy the equations

$$\begin{aligned} B^{(n)} \tau_\mu B^{(n)\dagger} &= \tau_\mu \quad (\mu=0, 1, \dots, n-1), \\ B^{(n)} \tau_n B^{(n)\dagger} &= -\tau_n, \end{aligned} \quad (18)$$

which are necessary and sufficient conditions for these $B^{(n)}$ to correspond to i_n .

For odd n , on the other hand, because of

$$\tau_n = \lambda_n \tau_1 \tau_2 \dots \tau_{n-1},$$

⁵ The existence of a unimodular matrix $B^{(n)}$ corresponding to i_n for even n for a given irreducible set $\{\tau_k\}$ ($k=1, 2, \dots, n$) can also be established as follows. For every n of this type, we adjoin to this set the matrix $\tau_{n+1} = \lambda_{n+1} \tau_1 \tau_2 \dots \tau_n$ and hence obtain an irreducible set $\{\tau_k\}$ ($k=1, 2, \dots, n+1$) of $n+1$ matrices of dimension $2^{n/2}$ fulfilling (1). [Plainly, (1) would not be obeyed by all these $n+1$ matrices if n had been assumed to be odd.] Now, we know that there exists for each even n a $2^{n/2}$ dimensional, unimodular matrix $B^{(n)}$ satisfying (5) for this last set of τ_k in the case of the restricted LORENTZ transformation (spatial rotation) in $n+1$ spacelike dimensions given by $x'^\mu = x^\mu$ ($\mu=1, 2, \dots, n-1$), $x'^n = -x^n$, $x'^{n+1} = -x^{n+1}$. Such $B^{(n)}$ are immediately seen to obey (18) and therefore to correspond to i_n .

there can be no $B^{(n)}$ to satisfy (18). The first equation (18), for $\mu=0$, shows that $B^{(n)}$ would have to be unitary. This unitarity and the rest of the same equation shows that $B^{(n)}$ would have to commute with all $\tau_1, \dots, \tau_{n-1}$. It cannot anticommute then with $\tau_n = \lambda_n \tau_1 \tau_2 \dots \tau_{n-1}$.

We complete the proof of Theorem 1 by showing that, if the unimodular matrices B_1 and B_2 correspond to the same LORENTZ matrix \mathcal{A} , then B_1 and B_2 are equal to within a factor of an N^{th} root of unity. Our method of proving this assertion is parallel to the one applied in reference ² to the special case $n=3$.

One easily finds from (5) and the nonsingularity of B_1 and B_2 :

$$B_1^{-1} B_2 \tau_\mu = \tau_\mu B_1^\dagger B_2^{\dagger-1} \quad (\mu=0, 1, \dots, n). \quad (19)$$

Setting $\mu=0$ in (19), we infer that

$$B_1^{-1} B_2 = B_1^\dagger B_2^{\dagger-1}$$

and hence that

$$B_1^{-1} B_2 \tau_k = \tau_k B_1^{-1} B_2 \quad (k=1, 2, \dots, n). \quad (19')$$

However, as was proved in reference ⁴, the irreducible set $\{\tau_k\}$ ($k=1, 2, \dots, n$) generates an irreducible representation of a certain finite abstract group. Therefore, (19') entails that $B_1^{-1} B_2$ commutes with all the matrices of this representation and thus that it is a multiple ω 1 of the $N \times N$ unit matrix, i. e., that

$$B_2 = \omega B_1. \quad (20)$$

Taking the determinant of both sides of (20), the unimodularity of B_1 and B_2 yields that $\omega^N = 1$, so that B_1 and B_2 are related in the stated manner.

Proof of Theorem 2.

We first dispose of the easy matters of the uniqueness and reality of the matrix $L \equiv \|L^\mu_\nu\|$ in (5'). The first property follows immediately from the linear independence of the matrices τ_μ . The reality of L can be proved by taking the hermitean adjoint of both sides of (5') and using the equation $\tau_\mu^\dagger = \tau_\mu$, thus obtaining

$$\sum_\mu \bar{L}^\mu_\nu \tau_\mu = \sum_\mu L^\mu_\nu \tau_\mu,$$

where $\bar{}$ means complex conjugation. Hence $\bar{L}^\mu_\nu = L^\mu_\nu$, because of the linear independence of the τ_μ . Incidentally, for each n , the hermiticity of the τ_μ can also be shown to be necessary to insure the reality

of the matrix L corresponding to each unimodular N -dimensional matrix B .

Lemma 1. *The matrix $X(y) = \sum y^\mu \tau_\mu$ is positive (negative) definite if and only if the $(n+1)$ -vector y is timelike and $y^0 > 0$ ($y^0 < 0$). It is nonsingular and indefinite if and only if y is spacelike. It is semidefinite if and only if y is a null vector.*

Proof. If y is timelike it can be written as $y = A' \cdot (a, 0, 0, \dots, 0)$, where $a \neq 0$ has the same sign as y^0 and A' is a restricted LORENTZ matrix. Hence, by Theorem 1, there exists a unimodular $B(A')$ such that

$$\begin{aligned} X(y) &= B(A') X((a, 0, 0, \dots, 0)) B(A')^\dagger \\ &= B(A') (a \tau_0) B(A')^\dagger = a B(A') B(A')^\dagger, \end{aligned} \quad (21a)$$

so that $X(y)$ is positive (negative) definite if a , and therefore y^0 , is positive (negative).

If y is spacelike, there is a restricted LORENTZ matrix A'' such that $y = A'' \cdot (0, b, 0, \dots, 0)$, and therefore a unimodular matrix $B(A'')$ in terms of which

$$\begin{aligned} X(y) &= B(A'') X((0, b, 0, \dots, 0)) B(A'')^\dagger \\ &= b B(A'') \tau_1 B(A'')^\dagger. \end{aligned} \quad (21b)$$

Because the characteristic values of τ_1 are 1 and -1 , (21 b) implies that $X(y)$ is nonsingular and indefinite for such spacelike y .

If y is a null vector, a restricted LORENTZ matrix A''' and a unimodular $B(A''')$ can be found such that $y = A''' \cdot (c, c, 0, \dots, 0)$. Therefore,

$$\begin{aligned} X(y) &= B(A''') X((c, c, 0, \dots, 0)) B(A''')^\dagger \\ &= c B(A''') (\tau_0 + \tau_1) B(A''')^\dagger \end{aligned} \quad (21c)$$

in this case. Now, the characteristic values of $\tau_0 + \tau_1$ are 0 and 2, so that (21 c) informs us that $X(y)$ is semidefinite for each null vector y . One can prove this property of $\tau_0 + \tau_1$ by first observing that (1) implies that $(\tau_0 + \tau_1)^2 = 2(\tau_0 + \tau_1)$, whence $\tau_0 + \tau_1$ has at least one of the numbers 0, 2 as characteristic values. Not all of the characteristic values of the matrix $\tau_0 + \tau_1$ are the same for, if they were, it would be a multiple of the unit matrix and would thus commute with all the τ_k , contrary to (1), so that both 0 and 2 are eigenvalues of $\tau_0 + \tau_1$.

Lemma 2. *For each $(n+1)$ -vector y ,*

$$\det X(y) = (y^2)^{N/2}, \quad (22)$$

where $y^2 \equiv (y^0)^2 - (y^1)^2 - (y^2)^2 - \dots - (y^n)^2$. (23)

Proof. Suppose that y is spacelike. Then we infer from (21 b), the unimodularity of $B(A'')$ in this equation, (17), and (23), that

$$\begin{aligned} \det X(y) &= b^N \det \tau_1 = b^N (-1)^{N/2} = (-b^2)^{N/2} \\ &= (y^2)^{N/2}. \end{aligned}$$

A similar, simpler, proof yields (22) when y is timelike.

Proceeding as in the previous two cases, one obtains for each null vector y , employing (21 c) and the singular nature of $\tau_0 + \tau_1$:

$$\det X(y) = c^N \det(\tau_0 + \tau_1) = 0,$$

so that (22) also holds in this case.

Returning to Theorem 2, let B denote an arbitrary unimodular matrix for which there exists a matrix L such that (5') is fulfilled, so that the equation

$$B X(y) B^\dagger = X(Ly) \quad (24)$$

holds identically for this B and L . Applying Lemma 2 to (24), one concludes that

$$\begin{aligned} ((Ly)^2)^{N/2} &= (y^2)^{N/2}, \\ \text{that is, } (Ly)^2 &= \omega(L, y) y^2, \end{aligned} \quad (25)$$

where $\omega(L, y)$ is an $\frac{1}{2} N^{\text{th}}$ root of unity [Actually, $\omega(L, y) = \pm 1$ since L is a real matrix].

Eq. (25) entails trivially that

$$(Ly)^2 = y^2 \quad (26)$$

for any null vector y . We wish to show that (26) also obtains for each timelike and each spacelike y . By virtue of (25), this is equivalent to proving that $(Ly)^2$ is timelike or spacelike if y^2 has one of these respective properties.

Let us apply Lemma 1 to a timelike vector y having $y^0 > 0$. Then $X(y)$ is positive definite and so is $X(Ly) = B X(y) B^\dagger$, since B is nonsingular, which implies that Ly is timelike and $(Ly)^0 > 0$. Similarly, if y is timelike and $y^0 < 0$, we conclude that then Ly is timelike and $(Ly)^0 < 0$. The fact that Ly is spacelike when y has this last property is proved analogously.

Hence, we have shown that L is a real homogeneous LORENTZ transformation. Moreover, $L^0_0 > 0$, i. e., L is orthochronous, since $(Ly)^0$ has the same sign as y^0 for each timelike y .

It remains to be established that, for every odd $n \geq 3$, the matrix L corresponding in the sense of (5') to each unimodular $N \times N$ matrix B is a restricted LORENTZ matrix. This will certainly be the

case if there does not exist for any such n an $N \times N$ unimodular matrix which satisfies (5') when L is chosen to be the improper LORENTZ matrix i_n pertaining to the spatial reflection (15). This was, however, established after (18).

2. Generalization to $n \geq 2$ of a Theorem on the Unimodular Matrices Associated with the Transpose of a Lorentz Matrix

We shall present two proofs of the following theorem:

Theorem 3. One has

$$B(A^T) = \omega B(A)^\dagger \quad (27)$$

for every orthochronous or restricted $(n+1)$ -dimensional LORENTZ matrix A in the respective cases of even n or of odd $n \geq 3$, where ω is an N^{th} root of unity.

In these proofs and in all of the subsequent discussions, symbols such as ω_0 , ω_1 , ω_2 , ω' , ω'' will serve to denote N^{th} roots of unity.

First Proof

Since, if the unimodular matrices $B(A_1)$ and $B(A_2)$ correspond to the respective LORENTZ matrices A_1 , A_2 , then $B(A_1 A_2)$ corresponds to $A_1 A_2$, one finds from Theorem 1

$$B(A_1 A_2) = \omega_1 B(A_1) B(A_2). \quad (28 a)$$

$$\text{Similarly, } B(A^{-1}) = \omega_2 B(A)^{-1}. \quad (28 b)$$

If theorem 3 holds for A_1 and A_2 , it also holds for $A_1 A_2$. Indeed, its validity for A_1 and A_2 entails $B((A_1 A_2)^T) = B(A_2^T A_1^T) = \omega' B(A_2^T) B(A_1^T)$
 $= \omega'' B(A_2)^\dagger B(A_1)^\dagger$
 $= \omega'' (B(A_1) B(A_2))^\dagger$
 $= \omega''' B(A_1 A_2)^\dagger,$

where we have used (28 a) and elementary properties of the roots of unity. Hence (27) holds for the product $A = A_1 A_2, \dots, A_r$ of any finite number r of factors if it holds for each of these factors.

But each restricted or improper orthochronous LORENTZ matrix A can be expressed as a product (6) of two proper spatial rotations and a pure LORENTZ transformation A_k or as a product (14) of a spatial reflection matrix i_n and a restricted LORENTZ matrix, respectively. Hence it suffices to prove (27) for the case when A is equal to a proper spatial rotation R , or to A_k or i_n .

Let $A = R$. Applying (4) to the vector $x = (1, 0, 0, \dots, 0)$, or setting $v = 0$ in (5), we find that $x' = R x = x$ and $X(x) = X(x') = \tau_0 = 1$. Therefore

$$B(R) B(R)^\dagger = 1,$$

i. e., $B(R)$ is unitary. In view of this fact, the property $R^T = R^{-1}$, and (28 b),

$$B(R^T) = B(R^{-1}) = \omega_0 B(R)^{-1} = \omega_0 B(R)^\dagger,$$

so that (27) holds for such rotations.

If $A = A_k$ or $A = i_n$, then $A^T = A$. Moreover, the unimodular matrix B_k in (8) corresponding to A_k and the unimodular matrix $B^{(n)}$ in (16) pertaining to i_n for even n have the properties

$$B_k^\dagger = B_k \quad \text{and} \quad B^{(n)\dagger} = \pm B^{(n)},$$

where the $+$ ($-$) sign obtains if $n = 4m + 2$ ($n = 2$ or $n = 4m$), for $m = 1, 2, \dots$. Consequently, if $A = A_k$ or $A = i_n$, we infer by employing Theorem 1 that

$$B(A^T) = B(A) = \omega_0' B(A)^\dagger,$$

so that (27) also obtains for these two types of A . Therefore (27) is valid for arbitrary orthochronous (restricted) A for even n (odd $n \geq 3$).

Second Proof

This proof is based on the equation

$$X(Gx) = x^2 X(x)^{-1}, \quad (29)$$

where G is the diagonal matrix in (3) with diagonal elements $1, -1, \dots, -1$ and x^2 is defined in (23). Eq. (29) holds for any $(n+1)$ -vector x for which $x^2 \neq 0$.

In order to prove that such x satisfy this equation, we employ (1), (2), and the equations $g_{00} = 1$, $g_{kk} = -1$ to write

$$\begin{aligned} X(x) X(Gx) &= \left(\sum_\mu x^\mu \tau_\mu \right) \left(\sum_\nu g_{\nu\nu} x^\nu \tau_\nu \right) \\ &= \frac{1}{2} \sum_{\mu, \nu} x^\mu x^\nu (g_{\nu\nu} \tau_\mu \tau_\nu + g_{\mu\mu} \tau_\nu \tau_\mu) \\ &= \frac{1}{2} \sum_{k, l} x^k x^l (-\tau_k \tau_l - \tau_l \tau_k) \\ &\quad + \frac{1}{2} \sum_k x^k x^0 (\tau_k \tau_0 - \tau_0 \tau_k) \\ &\quad + \frac{1}{2} \sum_l x^0 x^l (-\tau_l \tau_0 + \tau_0 \tau_l) \\ &\quad + \frac{1}{2} (x^0)^2 (\tau_0^2 + \tau_0^2) \\ &= \left[-\sum_k (x^k)^2 + (x^0)^2 \right] 1 = x^2 1. \end{aligned}$$

Therefore, if $x^2 \neq 0$, $X(x)^{-1}$ exists and obeys (29).

To prove Theorem 3, we begin by using (3) and the property $G^2 = 1$ to write

$$A^T = G A^{-1} G. \quad (3')$$

We also observe that, since A^{-1} and G are LORENTZ transformations,

$$(A^{-1} G x)^2 = (G x)^2 = x^2, \quad (30)$$

so that, in particular, $A^{-1} G x$ and $G x$ are not null vectors if x is not a null vector.

Let A pertain to one of the LORENTZ groups specified in Theorem 3 and let $x^2 \neq 0$. We then obtain, by combining (29) with (3'), (4), (28 b), and (30):

$$\begin{aligned} X(A^T x) &= X(G(A^{-1} G x)) \\ &= (A^{-1} G x)^2 X(A^{-1} G x)^{-1} \\ &= x^2 [B(A^{-1}) X(G x) B(A^{-1})^\dagger]^{-1} \\ &= x^2 [B(A)^{-1} \cdot x^2 X(x)^{-1} \cdot B(A)^{\dagger-1}]^{-1} \\ &= x^2 B(A)^\dagger X(x) (B(A)^\dagger)^\dagger. \end{aligned} \quad (31)$$

Now, as is clear geometrically and easy to establish algebraically, each null $(n+1)$ -vector can be written as the sum of two $(n+1)$ -vectors which are not null. Hence, because of the linearity of $X(x)$ in x , (31) is also valid when $x^2 = 0$, and thus holds for all $(n+1)$ -vectors x ⁶. We therefore conclude that $B(A)^\dagger$ corresponds to A^T , and therefore that it differs at most by a factor of an N^{th} root of unity from $B(A^T)$.

It may be of interest to point out that (27) allows one to prove very rapidly that $B(A)$ is unitary of and only if A is a spatial rotation, and that a necessary and sufficient condition for $B(A)$ to be hermitean or skew hermitean is that A be symmetric.

In conclusion, we shall prove that for $n=2, 3$ it is impossible to select a $B(A)$ so that (27) obtains with the $+$ sign for all relevant A , but that for $n>3$ one can make such a selection.

If B is a unimodular matrix corresponding to A , then B^\dagger corresponds to A^T by Theorem 3. Hence the most general unimodular matrices $B(A)$ and $B(A^T)$ corresponding to A and A^T are

$$B(A) = \omega' B, \quad (32 a)$$

$$B(A^T) = \omega'' B^\dagger \quad (32 b)$$

and therefore

$$B(A^T) = \omega'' B^\dagger = \omega'' (\bar{\omega}' B(A))^\dagger = \omega' \omega'' B(A)^\dagger.$$

⁶ The fact that (31) holds for null $(n+1)$ -vectors x can also be proved by a continuity argument.

If $A^T \neq A$, ω' and ω'' can be chosen independently, and therefore (27) can be made to hold with $\omega = 1$ for each $n \geq 1$ for such A .

Let us now investigate the case when $A = A^T$. If B is an arbitrary unimodular matrix corresponding to a A of this type, then clearly

$$B^\dagger = \omega''' B. \quad (33)$$

If A is any LORENTZ transformation, symmetric or not, such that (5) has unimodular solutions corresponding to it then among these solutions there is one of the form

$$B = \sum_A c_A \tau_A, \quad (34)$$

where the summation runs over a set of linearly independent matrices τ_A which are products of the τ_k and which therefore have the property $\tau_A^\dagger = \pm \tau_A$, and all the coefficients c_A are real and are evidently not all zero. One can prove (34) by combining the fact that a restricted or improper orthochronous A can be written in one of the respective forms (6) or (14) with (8) and with the simple result that to each proper spatial rotation there corresponds a unimodular matrix which is the product of matrices of the type (13). From Eqs. (33) and (34),

$$\sum_A \pm c_A \tau_A = \omega''' \sum_A c_A \tau_A,$$

whence $\omega''' = 1$, so that (33) becomes

$$B^\dagger = \pm B \quad (33')$$

for a B of the form (34). Because of (32 a) and (33'), the most general matrix $B(A)$ corresponding to a symmetric A obeys

$$\begin{aligned} B(A^T) &= B(A) = \omega' B = \pm \omega' B^\dagger \\ &= \pm \omega' (\bar{\omega}' B(A))^\dagger = \pm (\omega')^2 B(A)^\dagger. \end{aligned} \quad (35)$$

Let A be a spatial rotation (12) in the $x^k - x^l$ plane with $\varphi = \pi$. Then $A^T = A$ and the unimodular matrix $B = \tau_k \tau_l = -B^\dagger$ corresponds to this A . Hence (39) obtains with a $-$ sign for this A . Suppose now that $n=2, 3$. Then $N=2$, so that $(\omega')^2 = 1$, and therefore (35) yields $B(A^T) = -B(A)^\dagger$ for the rotation in question, no matter how $B(A)$ is chosen.

If $N \geq 4$, N is divisible by 4, and hence i is an N^{th} root of unity in this case. Therefore we can always choose ω' in (35) for any n of this type so that $\pm (\omega')^2 = 1$, i. e., so that (27) holds with $\omega = 1$ when $A^T = A$. Consequently, (27) can be made to hold with this value of ω for all A of interest when $n \geq 4$.